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THE COMPLEXITY OF THE TRAJECTORIES OF A DYNAMICAL SYSTEM

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In this paper we define the notion of complexity of a trajectory of a dynamical system and study the connection of this new quantity with such traditional characteristics of the theory of dynamical systems (see for example, the survey [1]) as entropy (both metrical and topological) and decomposition into ergodic sets. A relationship between entropy and complexity of the description of the information of a discrete ergodic source was first mentioned by Shannon, true, without a precise definition of the second quantity. A rigorous definition of the complexity of a finite word S was given by Kolmogorov [2]. In [3] there are many examples pointing to the fruitfulness of introducing a characteristic like complexity in the theory of algorithms, probability, and information; Proposition 5.1 announced there by Levin was one of the stimuli of the present paper.

A *dynamical system* (d.s.) is a pair (X, T), where X is a Lebesgue space with a σ-algebra of measurable sets $\mathfrak{M}$, and the transformation $T\colon X \to X$ is measurable. Let $M(X)$ be the set of all normalized measures of X. We put

$$M(T) = \{\mu \in M(X) : \mu \text{ is } T\text{-invariant}\},$$
$$EM(T) = \{\mu \in M(T) : \mu \text{ is ergodic}\}.$$

Let $u = \{U_1, U_2, \ldots, U_k\}$ be a finite cover of X ($\bigcup_{i=1}^{k} U_i = X$), $\mathcal{L} = \{1, 2, \ldots, k\}$, $\Lambda = \prod_{i=0}^{\infty} \mathcal{L}_i$, where $\mathcal{L}_i = \mathcal{L}$ for $i \in Z^+ = \{0, 1, \ldots\}$. We define a map $\varphi_u\colon X \to \Lambda$ as follows: $\omega \in \varphi_u(x)$ if $\omega = \{\omega(i)\}_{i\in Z+}$ is such that $T^i x \in U_{\omega(i)}$ for $i \in Z^+$. Finally, ω^n denotes the word $\omega(0)\,\omega(1)\ldots\omega(n-1)$, and $K(S)$ the complexity of a finite word S, as introduced in [2].

DEFINITION 1. *The complexity of the trajectory of a point $x \in X$ relative to a finite measurable partition β is defined as the quantity*

$$K^0(x, T \mid \beta) = \limsup_{n\to\infty} (1/n)\, K((\varphi_\beta(x))^n).$$

For all $x \in X$ the following relations hold:

1. $K^0(x, T \mid \beta_1) \leqslant K^0(x, T \mid \beta_2)$ if $\beta_1 \leqslant \beta_2$,
2. $K^0(x, T \mid \beta_1 \vee \beta_2) \leqslant K^0(x, T \mid \beta_1) + K^0(x, T \mid \beta_2)$;
3. $K^0(x, T \mid \beta^n) = K^0(x, T \mid \beta)$ for all $n \in Z^+$.

LEMMA 1 (the main lemma). *Suppose that $\mu \in EM(T)$. Then $K^0(x, T \mid \beta) = h_u(T, \beta)$ for μ-almost all x, where $h_\mu(T, \beta)$ is the entropy on the unit of time of β.*

DEFINITION 2. The *complexity of the trajectory of a point x relative to an exhaustive family of increasing partitions* $\mathfrak{B} = \{\beta_i\}_{i\in Z+}$ (that is, $\beta_0 \leqslant \beta_1 \leqslant \ldots$ and $\bigvee_{i\in Z+} \mathfrak{M}(\beta_i) = \mathfrak{M}$) is defined as the quantity

$$K^0(x, T \mid \mathfrak{B}) = \lim_{i\to\infty} K^0(x, T \mid \beta_i).$$

It can be shown that if $\mathfrak{B} = \{\beta_i\}$ and $\mathfrak{B} = \{\beta'_i\}$ are two distinct such families, then $Q = \{x\colon K^0(x, T \mid \mathfrak{B}) \neq K^0(x, T \mid \mathfrak{B}')\}$ is a set of absolute invariant measure zero. It is, therefore, expedient to fix some family $\mathfrak{B}$, to define the complexity of a trajectory as that relative to $\mathfrak{B}$, and to denote it simply by $K^0(x, T)$.

THEOREM 1. *For every $t \in R^+$ we set $X_t^0 = \{x\colon K^0(x, T) = t\}$; clearly, $X = \bigcup_{t\in R^+} X_t^0$.*

1. *The sets X_t^0 are invariant and measurable for all $t \in R^+$.*
2. $X_t^0 \cap X_{t'}^0 = \varnothing$ *for $t \neq t'$.*

3. *If* $\mu \in EM(T)$ *and the entropy of the d.s. is* $h_\mu(T)$ *then* $\mu\,(X_t^0) = 1$.

COROLLARY. *Let* $r = \sup_{\mu \in EM(T)} h_\mu$; *then* $R = \{x\colon K^0(x, T) > r\}$ *is a set of absolute invariant measure zero.*

However, the set R may be non-empty. In the case of a topological d.s. (for which $r = h(T)$ is the topological entropy) this circumstance compels us to consider also another definition of the complexity of a trajectory. In what follows it is assumed that X is a compact, possibly metric, space, $\mathfrak{M}$ a Borel σ-algebra, and T continuous.

DEFINITION 3. The quantity $K^1(x, T \mid u) = \limsup_{n \to \infty} (1/n) \left(\min_{\omega^n \in (\varphi_u(x))^n} K(\omega^n) \right)$ is called the *regular complexity of the trajectory of a point x relative to a finite cover u*, and the quantity $K^1(x, T) = \sup_u K^1(x, T \mid u)$ the *regular complexity of the trajectory of x*, where the sup is taken over all finite open covers u.

For $K^1(x, T \mid u)$ and $K^1(x, T)$ the following relations hold:

1′–3′. For $K^1(x, T \mid u)$ the assertions 1.–3. are valid.

4. $K^1(x, T \mid u) \leqslant h(T, u)$, the topological entropy of the cover u.

Consequently, $K^1(x, T) \leqslant h(T \mid O(x)) \leqslant h(T)$ for all x, where $\bar{O}(x)$ is the closure of the trajectory of x.

5. $K^1(x, T) \leqslant K^0(x, T \mid \mathfrak{B} = \{\beta_i\})$ for all x when the diameter of the partitions β_i tends to zero.

By means of $K^1(x, T)$ we can construct a partition of X analogous to that furnished by Theorem 1, since Theorem 2 holds.

THEOREM 2. *If* $\mu \in EM(T)$, *then* $K^1(x, T) = h_\mu(T)$ *for* μ*-almost all* x.

COROLLARY. *If* $X = \cup G_\mu \cup G_0$ *is a partition into ergodic sets (see* [1]), $X_t^1 = \{x\colon K^1(x, T) = t\}$, *and* $W_t = \{\cup G_\mu\colon h_\mu(T) = t\}$, *then* $X_t^1 \,\Delta\, W_t$ *is a set of absolute invariant measure zero, possibly non-empty.*

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